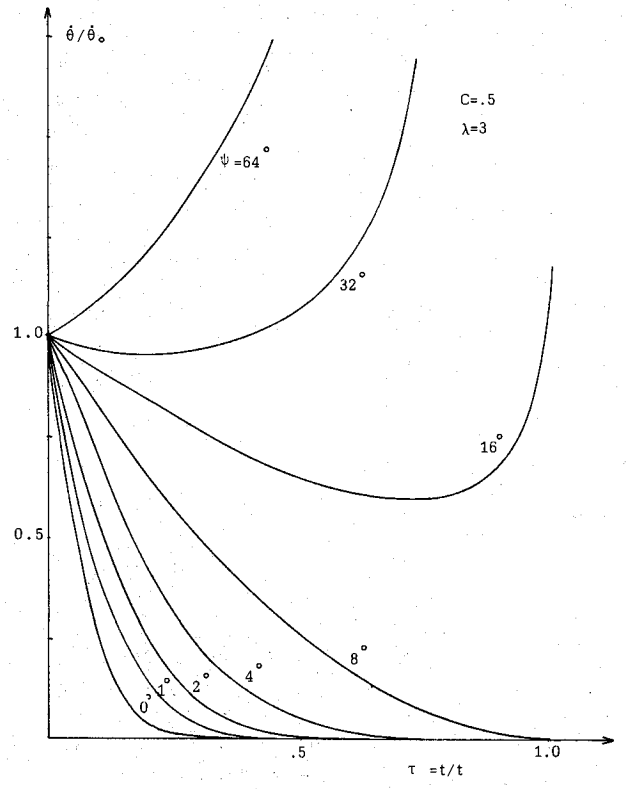

Fig. 3 Line-of-sight rotational rate vs time for various values of ψ.

reduced if ψ is increased from zero to a small positive value for a small value of c. The LOS angular rate $\dot{\theta} = H/r$ can be obtained from Eq. (13). The result is depicted in Fig. 3, where it can be seen that a large value of ψ results in an unbounded missile acceleration. It can also be shown[4] that the missile acceleration is always finite at the end of the pursuit if $m \leq \frac{1}{2}$.

From the several results considered above, it is suggested that GPN be used under conditions where c is small. A small value of c represents a small closing rate or a large target tangential velocity relative to the interceptor. Under these conditions, GPN has a larger capture area and a shorter interception time than TPN and the increment of the acceleration from that of TPN is small when ψ is kept small.

Conclusion

In this study, we have derived the closed-form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the generalized proportional navigation (GPN) law. Of great interest in this study is the fact that the analysis of the closed-form solution of GPN has enabled us to demonstrate the effectiveness of the concept.

A proper choice of ψ, the angle between the direction of interceptor acceleration and the direction normal to the line of sight, involves a tradeoff between the interceptor acceleration commanded and the interception time. Instead of a constant value of ψ, an optimal trajectory of ψ can be found if the weighted sum of these two factors is minimized.

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A New Method of Computing the State Transition Matrix for Linear Systems

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Introduction

A NEW method of computing the state transition matrix for a linear, time-invariant system is presented in this Note. Such a computation is required in estimation, filtering, and control problems where predictive computations are necessary.

Let a linear, time-invariant system be represented by

$$\dot{x}(t) = Fx(t) + Gu(t), \qquad t \in [t_0, t_f] \tag{1}$$

with initial condition $x(t_0) = x_0$, where the n-vector x denotes the state of the system, the p-vector u a forcing function, F and G the $n \times n$ continuous dynamics and the $n \times p$ forcing coupling matrices, respectively.

The solution[1,2] of Eq. (1) is obtained by means of the state transition matrix (or the fundamental matrix) $\Phi(t,t_0)$ for the homogeneous linear system corresponding to Eq. (1). For such a system, the transition matrix is just the exponential of the dynamics matrix F, the Peano-Baker formula.[3] The technique for computing the state transition matrix presented in this Note may be considered to be a new way of computing the exponential of a constant matrix. The problem of exponentiating a constant matrix has been studied extensively. Moler and van Loan[4] discuss at length 19 "dubious" ways of computing the exponential of a constant matrix. Reference 5 addresses the same problem and contains a good list of papers on various techniques for computing the exponential of a matrix.

Some discussion of the attributes of our technique is in order. It does not require a priori knowledge of the eigenvalues and eigenvectors of the dynamics (system) matrix. It is effective for a wide class of system matrices. It takes full advantage of the linear time-invariant nature of the system. Large transition intervals need not be divided into small transition intervals in order to preserve desired accuracy; our technique is valid for any finite transition interval.

We derive herein a compact formula for Φ for a consant matrix F, using what is known as the integral variation (IV) method.[6] We then employ this formula, the main result of this Note, to implement the prediction mode of a Kalman filter used for estimating the state (position and velocity) of a damped oscillator.

A Compact Formula for the Transition Matrix

Our method involves solving the homogeneous system corresponding to Eq. (1) and then obtaining the state transition matrix from the resulting solution. Hereafter, all vectors are column vectors.

Briefly, the IV method is a technique to generate an approximate solution of an initial value problem, whose original differential equations have been transformed into a system of

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first-order differential equations by directly solving a system of related integral equations, with the help of properly defined variation functions, to the first-order differential equations themselves.

We write the homogeneous system [Eq. (1)] as

$$x_a'(t) = \Delta t \sum_{b=1}^{n} f_{ab} x_b(\tau) \qquad (a=1,\ldots,n) \tag{2}$$

where f_{ab} are the elements of F, $\Delta t = t_f - t_0$, $\tau = (t-t_0)/(t_f-t_0)$; this time transformation $(t_0 \to \tau)$ maps $[t_0, t_f]$ into $[0,1]$. The prime denotes $\mathrm{d}/\mathrm{d}\tau$.

Introducing a set of n arbitrary, $C^r (r \geq 1)$ functions $h_a(\tau)$, known as variations or variation functions, satisfying $h_a(0) = 0$, we construct the n functions

$$\phi_a(\tau) \triangleq x_a(\tau) h_a'(\tau) + \Delta t \sum_{b=1}^{n} f_{ab} x_b(\tau) h_a(\tau) \tag{3}$$

According to the IV method, we do not solve Eq. (2) directly; rather, we solve the following integral equations

$$\int_0^1 \phi_a(\tau)\mathrm{d}\tau - x_a h_a \Big]_0^1 = 0 \qquad (a=1,\ldots,n) \tag{4}$$

The systems of Eqs. (2) and (4) are indeed equivalent: A solution of Eq. (2) is also the solution of Eq. (4), and conversely.[7]

To effect the integrations in Eq. (4), we consider generalized Fourier expansions for x_a and h_a, using the shifted Legendre polynomials[8] $P_i^*(\tau)$,

$$x_a(\tau) = x_{a0} + \sum_{i=1}^{N} \lambda_a^i Q_i(\tau) \qquad (x_{a0} = x_a(0))$$

$$h_a(\tau) = \sum_{j=1}^{N} \eta_a^j Q_j(\tau) \qquad (a=1,\ldots,n) \tag{5}$$

where

$$Q_i(\tau) \triangleq \sigma_i + P_i^*(\tau), \qquad Q_i(0) = 0, \qquad \sigma_i \triangleq (-1)^{i+1}$$

In Eq. (5), the N unknown coefficients λ_a^i are associated with x_a and η_a^j with h_a; N, the highest degree of the polynomials used, is increased with Δt. Parenthetically, any convenient set of orthogonal shifted polynomials may be used, such as shifted Chebyshev polynomials.

Substituting Eqs. (5) into Eq. (4) and performing the integrations yields

$$\sum_{j=1}^{N} \eta_a^j \sum_{i=1}^{N} \sum_{b=1}^{n} (\mu_{ji}\delta_{ab} - \Delta t\, f_{ab}\zeta_{ji})\lambda_b^i = \sum_{j=1}^{N} \eta_a^j \sum_{b=1}^{n} \Delta t\, f_{ab} x_{b0} \sigma_j \tag{6}$$

where δ_{ab} is the usual Krönecker delta symbol,

$$\zeta_{ji} \triangleq \int_0^1 Q_j(\tau) Q_i(\tau)\mathrm{d}\tau = \sigma_j\sigma_i + \frac{\delta_{ji}}{2j+1}$$

and

$$\mu_{ji} \triangleq \int_0^1 Q_j(\tau) P_i'(\tau)\mathrm{d}\tau = \sigma_j A_1(\mathrm{i},0) + A_1(\mathrm{i},\mathrm{j})$$

with

$$A_1(r,s) \triangleq [1-(-1)^{r+s}]\theta(r-s-1)$$

$$\theta(p) \triangleq 1, \quad p \geq 0$$

$$\triangleq 0, \quad p < 0 \tag{7}$$

Equations (6) lead to the following system of N algebraic equations to be solved for the unknown coefficients:

$$\sum_{b=1}^{n} \sum_{i=1}^{N} (\mu_{ji}\delta_{ab} - \Delta t\, f_{ab}\zeta_{ji})\lambda_b^i = \Delta t \sum_{b=1}^{n} f_{ab} x_{b0} \sigma_j \triangleq \gamma_a^j \qquad (j=1,\ldots,N;\quad a=1,\ldots,N) \tag{8}$$

To write Eq. (8) in matrix from, we form two nN vectors $\Lambda = (\lambda_1 \ldots \lambda_n)$ and $\Gamma = (\gamma_1 \ldots \gamma_n)$, where the N-vector $\gamma_a = (\gamma_a^1 \ldots \gamma_a^N)$. We also construct an $nN \times nN$ matrix M as follows:

$$M = \begin{bmatrix} \mu - \Delta t f_{11}\zeta & -\Delta t f_{12}\zeta & \ldots & -\Delta t f_{1n}\zeta \\ -\Delta t f_{21}\zeta & \mu - \Delta t f_{22}\zeta & \ldots & -\Delta t f_{2n}\zeta \\ \vdots & \vdots & & \vdots \\ -\Delta t f_{n1}\zeta & \ldots & & \mu - \Delta t f_{nn}\zeta \end{bmatrix} \tag{9}$$

in which the elements μ_{ji} and ζ_{ji} of the $N \times N$ matrices μ and ζ, respectively, are given in Eqs. (6). Note that, for a fixed time interval Δt, M is a constant matrix.

The system of Eq. (8) can then be written as a linear matrix equation

$$M\Lambda = \Gamma \tag{10}$$

in which the vector Γ, by virtue of Eq. (8), explicitly contains the initial condition x_0. The matrix equation (10) is the ultimate result established by the IV method. We then solve Eq. (10) for the unknown coefficient vector Λ and obtain the solution of Eq. (2) via Eq. (5).

To facilitate a compact expression for the transition matrix, we introduce the following two matrices: 1) an $nN \times N$ matrix,

$$L = \Delta t \begin{bmatrix} \sigma & 0 & \ldots & 0 \\ 0 & \sigma & \ldots & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdot & \cdot & \sigma \end{bmatrix} \tag{11}$$

and 2) an $n \times nN$ matrix

$$\psi = \begin{bmatrix} Q^T & 0 & \ldots & 0 \\ 0 & Q^T & \ldots & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & & & Q^T \end{bmatrix} \tag{12}$$

in which the N-vector σ is defined by $\sigma = (\sigma_1 \ldots \sigma_N)$ and the N-vector Q by $Q = (Q_1 \ldots Q_N)$, where the components Q_j is given in Eq. (5).

The L and $\psi(\tau)$ matrices allow us to express Γ in Eq. (10) as $\Gamma = LFx_0$. It then follows from Eqs. (5) and (10) that the solution $x(\tau)$ is

$$x(\tau) = x_0 + \psi M^{-1} LFx_0 \tag{13}$$

from which

$$\Phi(\tau,0) = I + \psi M^{-1} LF \tag{14}$$

where I is the $n \times n$ identity matrix.

We summarize the development above in the following proposition:

For a constant matrix F,

$$\Phi(\tau,0)=e^{F\tau}=I+\psi M^{-1}LF$$

where M, L, and ψ are given in Eqs. (9), (11), and (12), respectively.

The formula in Eq. (14) for the state transition matrix for a constant F is valid for a wide class of matrices F and for any time interval—no time-step partitioning is necesary. A large time interval requires only a large N, that is, a large set of polynomials.

Kalman Filtering Application

We now apply the formula in Eq. (14) in the computation of the state transition matrix for estimating the state (position and velocity) of the damped oscillator and its state covariance matrix, given noisy discrete position measurements.

The state is a two-vector x whose components are position x_1 and velocity x_2; the constant dynamic matrix F is given by

$$F=\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{bmatrix} \tag{15}$$

where ω_0 is the natural frequency and β the damping parameter. We will assume no process noise. The initial condition $\hat{x}_0=(\hat{x}_{10},\hat{x}_{20})$ is prescribed together with its covariance matrix $P_0=E[(x_0-\hat{x}_0)\,(x-\hat{x}_0)^T]$, where $\hat{\ }$ denotes an estimated quantity. At time $t_k>t$, there is a scalar position measurement z_k, which is linearly related to the state by

$$z_k=H_kx_k+v_k \tag{16}$$

where $H_k=[1\ 0]$ is a 1×2 constant matrix and v_k a discrete Gaussian noise of zero mean and covariance R_k.

Let I_k denote the time interval $[t_k,t_{k+1}]$, $t\epsilon I_k$, whose length need not be uniform over k. Position measurements are available at both ends of I_k. We now invoke the time transformation $t\to\tau$, $\tau=(t-t_k)/(t_{k+1}-t_k)$, which maps I_k into $[0,1]$ for $k=0,1,\ldots$ As a result, the Kalman filtering equations are free of the usual subscript k and hereafter we work with the normalized time τ. For instance, $\Phi(1,0)$ is the proper notation for the state transition matrix over any interval I_k.

Two remarks are in order. First, if the time intervals I_k all have the same length, the Φ (1,0) need be computed only once and stored for subsequent use. The computation requires the highest degree N of the polynomials used, which will remain invariant throughout, and inverting the matrix M only once. Thus, the prediction equations are implemented efficiently. Second, if a measurement should become available at some time within the kth interval I_k, the same coefficient vector Λ, which has been already obtained for the entire interval I_k, can be used to compute the appropriate state transition matrix that will yield the state and its covariance matrix at that time. The usual updating process does not change. Thus, flexibility in implementing the Kalman filter equations is achieved.

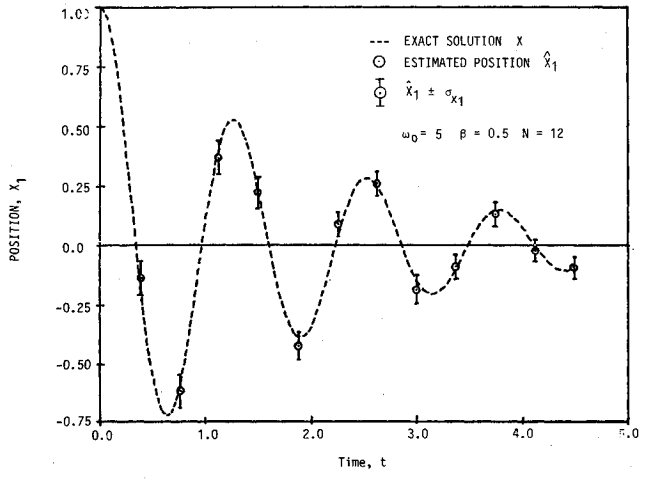

Fig. 1 Kalman filtering performance.

An explicit expression for the state transition matrix Φ (1,0) is now obtained for the damped oscillator. It follows from Eq. (14) that, setting $\tau=1$,

$$\Phi(1,0)=I+\psi(1)M^{-1}LF \tag{17}$$

The evaluation of $\psi(\tau)$ at $\tau=1$ is readily accomplished by means of Eqs. (5) and (12) and amounts to setting $Q_i(1)=\sigma_i+1$, since $P_i^*(1)=1$ for all values of i. The time interval in L in Eq. (12) now becomes Δt_k, the length of the time interval I_k.

It follows from Eq. (17) that the four elements of the transition matrix $\Phi(1,0)$ are

$$\begin{aligned}
\Phi(1,0)&=1-\Delta t_k\omega_0^2\sum_{i=1}^{N}\sum_{j=1}^{N}(\sigma_i+1)M^{-1}_{ij+N}\sigma_j\\
\Phi(1,0)&=\Delta t_k\sum_{i=1}^{N}\sum_{j=1}^{N}(\sigma_i+1)\,(M^{-1}_{ij}-2\beta M^{-1}_{ij+N})\sigma_j\\
\Phi(1,0)&=-\Delta t_k\omega_0^2\sum_{i=1}^{N}\sum_{j=1}^{N}(\sigma_i+1)M^{-1}_{i+Nj+N}\,\sigma_j\\
\Phi(1,0)&=1+\Delta t_k\sum_{i=1}^{N}\sum_{j=1}^{N}(\sigma_i+1)(M^{-1}_{i+Nj}-2\beta M^{-1}_{i+Nj+N})\sigma_j
\end{aligned} \tag{18}$$

Comparison of Results

To demonstrate the accuracy and reliability of our technique, we compare results from Eq. (18) with those obtained from using the exact expression for the state transition matrix of the damped oscillator. The comparison is made in two areas: the individual elements of the state transition matrix and the Kalman filtering performance. The exact state transition matrix $\Phi^*(1,0)$ is obtained directly from the exact solution of the damped oscillator.[9] It can be readily shown that $\Phi^*(1,0)$ is given by

$$\Phi^*(1,0)=\frac{e^{-\beta\Delta t_k}}{\omega_1}\times\begin{bmatrix} \omega_1\cos\omega_1\Delta t_k+\beta\sin\omega_1\Delta t_k & \sin\omega_1\Delta t_k \\ -(\omega_1^2+\beta^2)\sin\omega_1\Delta t_k & \omega_1\cos\omega_1\Delta t_k-\beta\sin\omega_1\Delta t_k \end{bmatrix} \tag{19}$$

in which $\omega_1^2\triangleq\omega_0^2-\beta^2$ and Δt_k is the length of the transition time interval I_k.

Table 1 contains the elements of the exact state transition matrix $\Phi^*(1,0)$ and the approximate matrix $\Phi(1,0)$ for three transition time intervals of length 1.5, 3.0, and 4.5 time units. For each element in Table 1, the top and bottom rows correspond to Φ^* and Φ, respectively. Here, the computation of Φ (1,0) involves $N=12$; the values of the damping parameter β and the damping-free characteristic frequency ω_0 are 0.5 and 5, respectively. Note the excellent agreement between Φ and Φ^* for the two smaller time intervals. Substantial differences appear, however, for the largest time interval 4.5, which corresponds to about 3.5 oscillations. As N is increased to 16, these differences can be virtually eliminated, as shown in Table 1; the maximum percentage error is 1.84% for the Φ_{12} element.

Table 1 Comparison of transition matrices ($N=12$)

Exact and approximate Φ_{ij}	Transition time interval Δt_k 1.5	3.0	4.5	
Φ_{11}	0.2242	−0.1423	−0.1013	
	0.2242	−0.1414	−0.1165	(−0.1019)[a]
Φ_{12}	0.0878	0.0316	−0.00817	
	0.0878	0.0316	−0.00159	(−0.00802)
Φ_{21}	−2.1940	−0.7911	0.2043	
	−2.1940	−0.7895	0.0398	(0.2006)
Φ_{22}	0.1364	−0.1740	−0.0932	
	0.1364	−0.1757	−0.1149	(−0.0938)

[a]Numbers in parentheses correspond to $N=16$.

The performance of the Kalman filter is illustrated in Fig. 1. A total time interval of 4.5 is used here; position measurements are given at $\Delta t=0.375$. The initial position estimate is $\hat{x}_0=1.0$ and its uncertainty is $\sigma_x=0.5$. Note the accuracy of the estimates obtained with the approximate Φ; virtually identical results (not shown) are obtained with Φ^*. Furthermore, the velocity variable $\hat{x}_2$ also shows a similar behavior.

Conclusions

In conclusion, we have developed a new technique, based on the integral variation method, for computing the state transition matrix for a time-invariant linear system and demonstrated its use in Kalman filtering. Finally, an analytico-numerical technique[10] (also connected with the integral variation method) for computing the state transition matrix for nonlinear systems has been developed and successfully demonstrated with two problems: the two-body problem of celestial mechanics and the powered flight of a missile.

Acknowledgment

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